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Damage Spreading in a 2D Ising Model with Swendsen–Wang Dynamics

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Damage spreading for Ising cluster dynamics is investigated numerically by using random numbers in a way that conforms with the notion of submitting the two evolving replicas to the same thermal noise. Two damage spreading transitions are found; damage does not spread either at low or high temperatures. We determine some critical exponents at the high-temperature transition point, which seem consistent with directed percolation.

KEY WORDS: Damage spreading; cluster algorithms.

Damage spreading[1–3] turned out to be a useful tool[4] to investigate the dynamics of Ising models. Two replicas of the same system, which initially differ only on a small subset of the lattice sites, are simulated using the same random numbers and one observes how this "damage" spreads during dynamics by a site-by-site comparison of the two replicas.

Research done[5,6] using the Metropolis, Glauber and Heat Bath methods, flipping locally one spin at a time, revealed that whether damage spreads for a particular model or not depends on the kind of dynamics being used. As was recently demonstrated,[7] this "subjective" aspect of damage spreading can be overcome if one considers all possible single-spin-flip dynamic procedures that are consistent with the physics of a single replica. The family of possible dynamic processes that satisfy this requirement is quite large; the above-mentioned methods form a small subset of this family. When more general processes were considered, damage was

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shown to spread[8] even for the one-dimensional Ising model, with exponents that were either in the Directed Percolation[9] (DP) or the Parity Conserving[10] universality classes.

In this publication we extend further the dynamic procedures for which damage spreading is defined. Whereas all work mentioned above was done for single-spin-flip dynamics, we study here evolution when a non-local procedure, the Swendsen–Wang (SW) cluster algorithm[11] is used. To our knowledge this was done, so far, only in ref. 12, which pointed out a conceptual difficulty in extending to SW the definition of "using the same random numbers on the two replicas." In the current paper we present one possible way to overcome this difficulty and address the issue of how damage spreads when a non-local algorithm is used and estimate the associated exponents.

To understand the difficulty mentioned above note that the SW algorithm consists of two steps in which random numbers are generated, namely the construction of the clusters and the assignment of their new orientation. In DS simulations, however, even when we use the same random numbers to generate the SW clusters, we will in general generate *different clusters* on the two replicas. Reference 12 attempted to associate clusters of one replica with those of the other by the order of the clusters' appearance and assigned the same random number to each such pair of clusters. Even the number of clusters in the two replicas is not the same; identification by order may well cause two groups of spins at very remote location being assigned the same random numbers. Hence the observation[12] that after many iterations the two replicas became quite uncorrelated can be attributed to the fact that the two replicas were, in fact, not submitted to the same thermal noise. (This problem can be avoided for the Wolff algorithm[11] where our tests gave always spreading of damage above T_c.)

We propose here another way, one which is more in the spirit of the standard definition of damage spreading, to deal with the random number problem for Swendsen–Wang damage dynamics. We study how damage spreads for the Ising model and present numerical results for $L \times L$ square lattices.

Our method works as follows. The first step of the SW procedure starts from a spin configuration and generates clusters. We do this by assigning a random number $0 \leqslant p_{ij} \leqslant 1$ to every *bond*, i.e., the same number is assigned to a given bond on the two replicas. Each bond is either frozen or deleted according to the standard SW rule:[11]

If $S_i S_j = -1$ the bond is deleted

If $S_i S_j = 1$ it is deleted if $p_{ij} \leqslant \exp(-2J/k_B T)$ and frozen otherwise

Sites connected by frozen bonds form the SW clusters

Clearly, identical spin configurations on the two replicas give rise to identical clusters.

The second step of the SW procedure assigns each cluster a spin value, ± 1, at random; i.e., by assigning a random number to each cluster. The problem mentioned above arises at this step; the number of clusters may differ in the two replicas, any spin can belong to one cluster in one replica and to a different one in the other. What is the meaning, in this situation, of using the same random number on the two replicas? To overcome this ambiguity, we assign different random numbers p_i to every site of the lattice, but the same random number is assigned to the same site i on the two replicas. The p_i are uniformly distributed about zero and the new status of every cluster is determined depending on whether the *sum of all the random numbers assigned to its sites* is positive or negative. Clearly, if two clusters in the two replicas contain identical sites, these sums will be identical, and the two clusters will be treated in the same way. If the two clusters are nearly identical, then the two sums will be strongly correlated and most probably treated the same way. If the two clusters share no common site, the two sums will be completely uncorrelated and so will be the cluster orientations. This completes the description of a single SW step, which is our Monte Carlo time unit. New random numbers are chosen after every step.[5]

The procedure outlined above is the analogue of the Heat Bath algorithm, to which we restrict our attention in this paper. After equilibration we introduce damage, flipping in one replica sites that belong to one line in the center of the lattice, and only the odd sites on it are damaged to prevent the "infinite" cluster from splitting into two halves below the critical temperature T_c. (We used helical boundaries in one and free boundaries in the other direction. We also initially damaged the center quarter of the whole lattice, and got the same spreading temperature as given below.)

Figure 1a shows the equilibrium damage as a function of temperature. As expected, damage does not spread at low temperatures; it starts to spread below T_c and its limiting long-time value reaches a maximum at T_c. For $T \leqslant T_c$ we find enormous fluctuations, and even extended regions of damage may vanish completely within a single time step. We ascribe this to the presence of an "infinite" cluster below T_c, which we flip just as we do the many finite clusters. Interestingly, there is a second transition—damage shrinks and vanishes above a spreading temperature T_s, with $1.33 < T_s/T_c < 1.34$ for $200 \leqslant L \leqslant 1000$. Close to T_s the damage seems to vanish as $(T_s - T)^\beta$ with $\beta = 0.65$; see Fig. 1b.

[5] It is natural to associate the random numbers with thermal noise which, in turn, is normally assumed to be local in space and time. In this sense our proposed procedure is more "physical" than the one used in ref. 12.

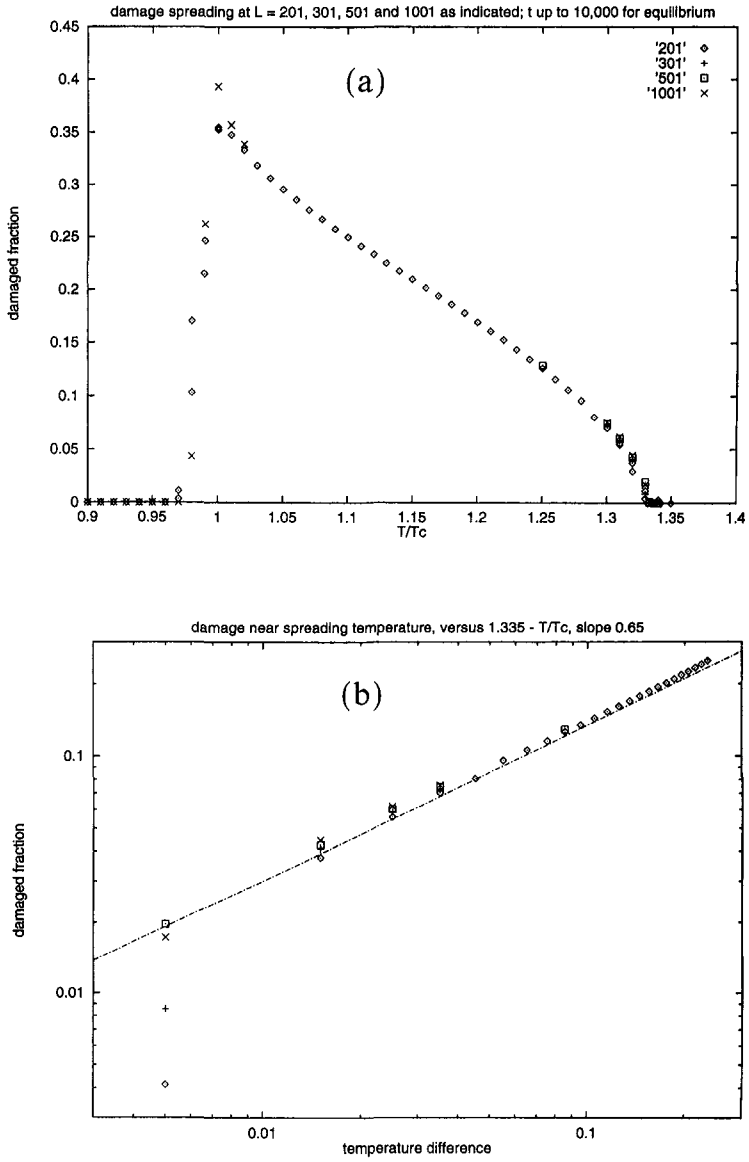

Fig. 1. Fraction of damaged sites in equilibrium versus temperature. Part a shows the whole investigated region on linear scales, part b gives double-logarithmically part of the same data, slightly below the spreading temperature.

The Swendsen–Wang dynamics was invented to reduce critical slowing down; indeed, right at $T=T_c$ the relaxation time in two dimensions increases only logarithmically with system size.[13] Our damage, however, shows strong critical slowing down on both sides of the spreading temperature T_s. For a quantitative study we use the damage vanishing method:[4, 14] Initially half of the lattice is damaged (if all of it is damaged, the damage vanishes immediately). Next, we checked for $T > T_s$ how the damage decays to zero; Fig. 2a shows that it does so exponentially, after an initial transient. Fig. 2b shows that these exponential relaxation times roughly equal $\tau_r \propto (T/T_c - 1.335)^{-1.2}$. When, however, instead of τ_r we studied τ_1, the average time after which the damage has become exactly zero (not shown), an exponent nearer to 0.9 was observed; such discrepancies have been observed earlier[14] with the latter definition of relaxation times.

We also performed more limited studies of damage spreading from a single initial site. In this type of simulation[15] one usually expects the damage to survive after t iterations with a probability $\propto t^{-\delta}$, the number of damaged sites to grow as $t^{\eta+\delta}$, and the mean square distance of the damaged region from the origin of the damage to grow as t^z. The latter two quantities are averaged only over those lattices which are still damaged at time t.

The initialization with a single seed of damage is, however, very inefficient numerically, since often the damage vanishes very fast and we end up simulating a whole lattice to study only a small region around this site. Hence to get meaningful results one must average the evolution over a very large number of sample runs which, in turn, limits the sizes of the lattices used. For example, we performed averages over 5000 runs of an 301×301 system; the results obtained this way may, therefore, be strongly influenced by finite size effects. With all these caveats taken into account we obtained (see Fig. 3) at $T=T_s$ the following results: a. The survival probability of damage to time t is $\propto 1/t^{0.5}$, i.e. we get $\delta = 0.5$ (versus the DP result[15] $\delta = 0.460(6)$); b. the number of damaged sites grows as $t^{0.7}$ (to be compared with $\delta + \eta = 0.681$ in DP—note that we measure damage per surviving runs); c. The mean square distance of the damaged region from the origin of the damage grows as $t^{1.1_5}$, whereas $z = 1.134$ in DP. Since our exponents (including our $\beta = 0.65$, vs 0.584) deviate from those of DP[15] by only about 10%, we believe that the observed damage spreading transition at $T=T_s$ belongs to the universality class of directed percolation.

In summary, we studied damage spreading using a non-local algorithm. By introducing a definition of damage spreading for the Swendsen–Wang algorithm which conforms with the standard notion of submitting the two replicas to the same thermal noise, in contrast with ref. 12, we

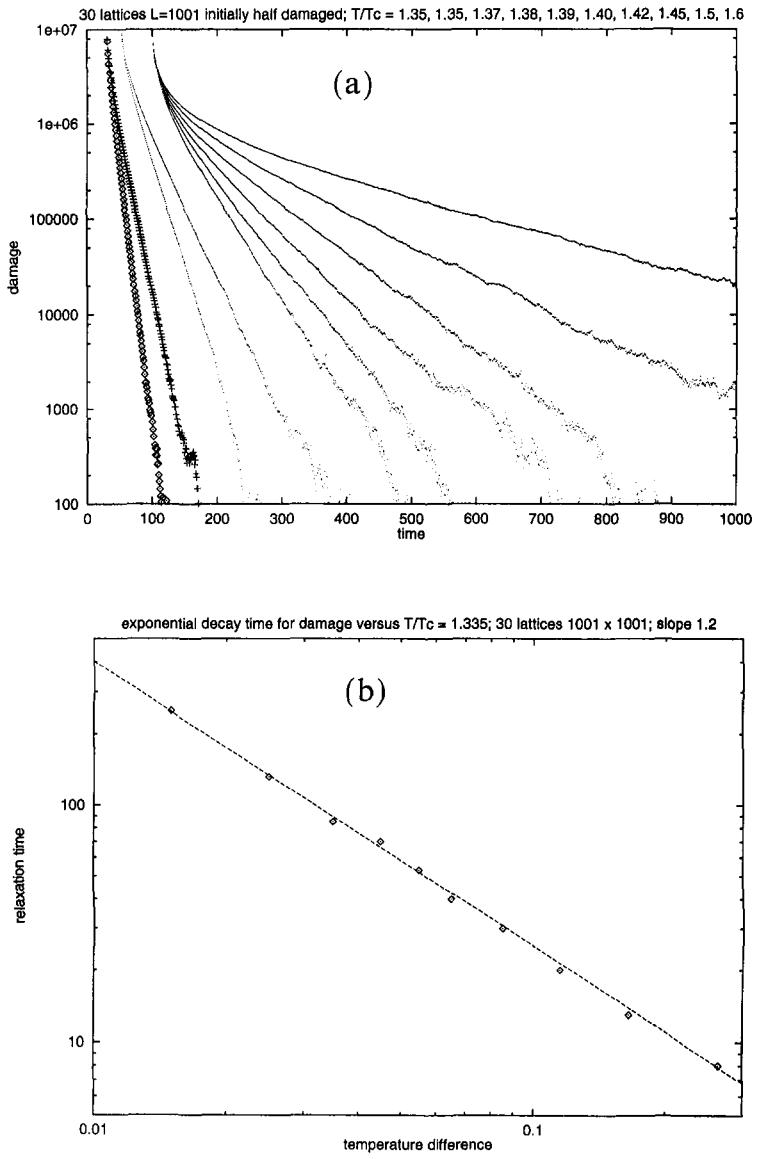

Fig. 2. Relaxation of the damage for $1.35 \leqslant T/T_c \leqslant 1.60$, that means above the spreading temperature; depending on the distance from T_s, damage is introduced at time = 30, 50, or 100. Straight lines in the semilogarithmic plots of part a correspond to $\exp(-t/\tau)$, and the relaxation times τ are plotted double-logarithmically in part b.

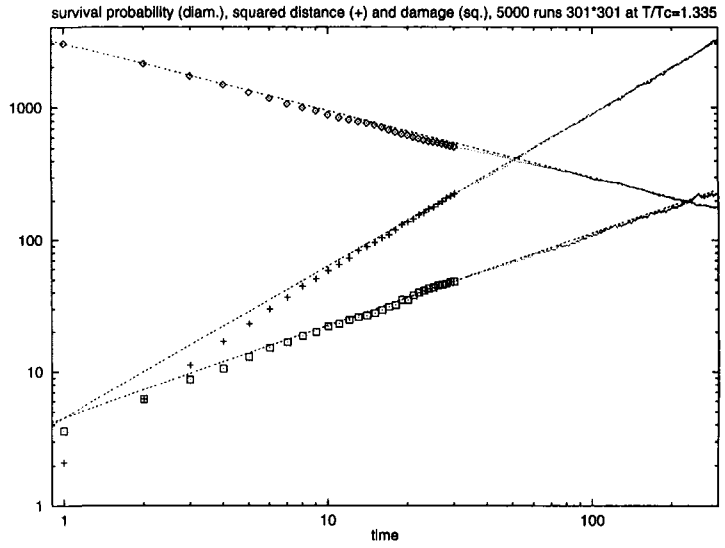

Fig. 3. Dynamics at the spreading temperature $T/T_c \simeq 1.335$: Number of surviving damages ($\diamond$) out of 5000 samples, mean square distance ($+$) and number of damaged sites ($\square$). The latter two quantities are summed up over all samples and then divided by the number of surviving damages ($\diamond$), i.e., by the number samples which were still damaged at that time t.

discovered some non-trivial results; damage spreads at temperatures *between* two transitions. Since at $T_s > T_c$ the clusters are finite the transition according to our numerical estimates of the exponents may well be in the standard DP universality class; below T_c, however, "infinite" clusters are present and the damage spreading transition may be in a new universality class. Possible extensions of our work may include working at other dimensions, with different spin models, as well as to improving the accuracy of our numerics on the square lattice.

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